On two Remarkable Pencils of Cubics of the Triangle Plane

Bernard Gibert

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Abstract

Let $X$ be a center\textsuperscript{1} in the plane of the reference triangle $ABC$. For any point $P$, denote by $X_a, X_b, X_c$ this same center in the triangles $PBC, PCA, PAB$ respectively. We seek the locus $E(X)$ of point $P$ such that the triangles $ABC$ and $X_aX_bX_c$ are perspective. The general case is a difficult problem and, in this paper, we only study the particular situation when $X$ is a center on the Euler line. We shall meet several interesting cubics.

1 The cubics $C(k)$

Let $k$ be a real number or $\infty$. Denote by $X$ the point such that $\overrightarrow{OX} = k \overrightarrow{OH}$ (where $O =$ circumcenter, $H =$ orthocenter) and denote by $X'$ the point on the Euler line defined by $\overrightarrow{OX'} = 1/k \overrightarrow{OH}$. $X'$ is clearly the harmonic conjugate of $X$ with respect to $H$ and $L = X_{20}$ (de Longchamps point).

1.1 A trivial case

When $X = G$ (centroid), the locus $E(G)$ is the entire plane. More precisely, for any $P$, the lines joining the vertices of $ABC$ to the centroids of the triangles $PBC, PCA, PAB$ concur at the complement of the complement of $P$. Hence, from now on, we take $X \neq G$ i.e. $k \neq 1/3$.

1.2 Theorem 1

For any center $X$ which is not $G$, the locus $E(X)$ is the union of the line at infinity, the circumcircle\textsuperscript{2} and a cubic curve denoted by $C(X)$.

Moreover, $C(X) = C(X')$ or equivalently $C(k) = C(1/k)$ (when $k = 3$, see \S 2.1 below). $C(X)$ and $C(X')$ pass through $A, B, C$, the reflections $A', B', C'$ of $A, B, C$ about the sidelines, $H, X$ and $X'$.

1.3 Theorem 2

All the cubics $C(X)$ obtained are in the same pencil of cubics which is generated by the Neuberg cubic $K_001$ (obtained for $X = O$ or $X = X_{30}$ i.e. $k = 0$ or $k = \infty$) and the union of the three altitudes ($X = H$ i.e. $k = 1$). Hence all these cubics pass through $A, B, C$, their reflections $A', B', C'$ in the sidelines of $ABC$ and $H$. $H$ being a triple point on the second (degenerated) cubic, all the cubics share the same tangent, the same polar conic (which is a rectangular hyperbola) and the same osculating circle at $H$ (see \S 4 for more details).

\textsuperscript{1}See [4], p.46
\textsuperscript{2}When $P$ lies on the circumcircle, the triangles $ABC$ and $X_aX_bX_c$ are homothetic at $Q$ which therefore also lies on a circle.
If we write an equation of the Neuberg cubic under the form
\[
N = \sum_{\text{cyclic}} (a^2 S_A - 2S_B S_C) x (c^2 y^2 - b^2 z^2) = 0
\]
and an equation of the union of the three altitudes as
\[
A = (S_A x - S_B y)(S_B y - S_C z)(S_C z - S_A x) = 0,
\]
a computation gives an equation of \(C(k)\), which rewrites as:
\[
(k - 1)^2 N - 4 k A = 0
\]

1.4 Theorem 3

- If \(k^2 - 3k + 1 \neq 0\), the locus of point \(M\) whose polar conic (with respect to \(C(k)\)) is a rectangular hyperbola is the Euler line. From this, there are three points (not always distinct when \(k = \pm 1\)) on \(C(k)\) in this situation: \(H, X\) and \(X'\).
- If \(k^2 - 3k + 1 = 0 \iff k = (3 \pm \sqrt{5})/2\), all the points in the plane have a polar conic which is a rectangular hyperbola. The corresponding cubic\(^3\) \(C(k)\) is called the ‘Golden Cubic’ and is studied in §2.3.

1.5 Theorem 4

\(C(X)\) meets the sidelines of \(ABC\) again at three points \(U, V, W\) which can be constructed as follows: if \(G_a\) is the projection of \(G\) on the altitude \(AH\) and if \(U_a\) is the barycentre of the system \(\{(A, k), (G_a, -3)\}\), then the line \(XU_a\) meets \(BC\) at \(U\).

Remark: when \(k = 3\), the barycentre is the point at infinity of the altitude \(AH\) and \(X\) is the reflection \(L_1\) of \(L\) about \(H\); in this case, \(UVW\) is the pedal triangle of \(L_1\) (see §2.1).

1.6 Theorem 5

The asymptotic directions of \(C(k)\) are those of \(pK(X6, Y_c)\) where the point \(Y_c\) is defined by
\[
\overrightarrow{OY_c} = \frac{k^2 - 3k + 1}{k} \overrightarrow{OH}.
\]

2 Some examples

2.1 The cubic \(K117 = C(3)\)

When \(k = 3\) i.e. \(X\) is the reflection \(L_1\) of \(L\) about \(H\)\(^4\), we obtain the cubic \(C(3)\) passing through \(G\), tangent to \(GK\) at \(G\), with a very simple equation:
\[
\sum_{\text{cyclic}} x \left[(3a^2 - 3b^2 + c^2)y^2 - (3a^2 + b^2 - 3c^2)z^2\right] = 0
\]

The homothety with center \(G\), ratio 4 transforms \(A, B, C\) into three points \(A_4, B_4, C_4\) lying on the curve. The asymptotes of \(C(3)\) are parallel to those of the Thomson cubic.

Remember also (see §1.5) that \(C(3)\) passes through the vertices \(U, V, W\) of the pedal triangle of \(L_1\). The homothetic of \(K\) (center \(G\), ratio 10) also lie on the curve: it is the coresidual of \(A, B, C, H\) and is collinear with \(U\) and \(A'\), \(V\) and \(B'\), \(W\) and \(C'\).

\(^3\)Observe here that we have two inverse values and therefore one single cubic.

\(^4\)This point has first barycentric coordinate \(3(b^2 - c^2)^2 + a^2(2b^2 + 2c^2 - 5a^2)\) and is not mentioned in [5].
2.2 The Soddy cubic $K_{032} = C(-1)$

When we take $k = -1$ or equivalently $X = X' = L$, we get a very interesting case $C(L)$ of such cubic and we shall call it the Soddy cubic since it passes through the eight Soddy centers. It is tangent at $L$ to the Euler line and meets the sidelines of $ABC$ on the lines passing through $L$ and the midpoints of the altitudes.

2.3 The Golden Cubic $K_{115}$

In §1.4, we had found $k = (3 \pm \sqrt{5})/2$. We associate the points $\Phi$ and $\Phi'$ (on the Euler line) to the real numbers $(3 + \sqrt{5})/2$ and $(3 - \sqrt{5})/2$ respectively. Hence,

$$\overrightarrow{OH} = \frac{3 + \sqrt{5}}{2}$$

which rewrites under the form

$$\overrightarrow{H} = \frac{1 + \sqrt{5}}{2}$$

We call $C(\Phi) = C(\Phi')$ the Golden Cubic which obviously passes through $A, B, C, A', B', C', H, \Phi$ and $\Phi'$. Its barycentric equation is:

$$\sum \text{cyclic } yz [y f(a, b, c) - z f(a, c, b)] = 0$$

where

$$f(a, b, c) = (b^2 - c^2)^3 + a^2 [b^2(c^2 + a^2 - 2b^2) + c^4].$$

Since all the polar conics are rectangular hyperbolas, $C(\Phi)$ has three real concurring asymptotes making $60^\circ$ angles with one another. They are perpendicular to the sidelines of the Morley triangle\(^5\) and concur at the point $Z$ intersection of the parallel at $G$ to the Brocard line $OK$ and the line $KX_{22}$, but this point does not lie on the curve. This type of cubic is called a $K_{060}^+$ in [1]. The first barycentric coordinate of $Z$ is

$$a^2(b^2c^2 + c^2a^2 + a^2b^2 - b^4 - c^4)$$

and this point is not mentioned in [5].

2.4 Other remarkable $C(k)$

2.4.1 The cubic $K_{116} = C(2) = C(1/2)$

$C(2)$ passes through $X_4, X_5, X_{382}$. See remark in §3.3 below.

2.4.2 The cubic $K_{119} = C(\Psi) = C(\Psi')$

With $k = (7 + \sqrt{33})/4$ and $k = (7 - \sqrt{33})/4$, we obtain the points $\Psi$ and $\Psi'$ on the Euler line. $C(\Psi)$ passes through $X_4, X_{17}, X_{18}, \Psi$ and $\Psi'$. Its asymptotes are parallel to those of the Napoleon cubic.

2.4.3 The cubic $K_{120} = C(\Theta) = C(\Theta')$

With $k = 2 + \sqrt{3}$ and $k = 2 - \sqrt{3}$, we obtain the points $\Theta$ and $\Theta'$ on the Euler line. $C(\Theta)$ passes through $X_4, \Theta$ and $\Theta'$. Its asymptotes are parallel to those of the orthocubic. The points $\Theta$ and $\Theta'$ are related to the isodynamic points $X_{15}, X_{16}$ and the Vecten points $X_{485}, X_{486}$ since we have the four following collinearities:

$$\Theta, X_{15}, X_{485} - \Theta, X_{16}, X_{486} - \Theta', X_{15}, X_{486} - \Theta', X_{16}, X_{485}$$

The circle with diameter $\Theta \Theta'$ is centered at $X_{382}$ and its radius is $\sqrt{3} OH$.\(^6\)

\(^5\)i.e. parallel to those of the McCay cubic.

\(^6\)X_{22} is the Exeter point.
3 Locus of the perspectors: the cubics $\mathcal{D}(k)$

In this paragraph we suppose $X$ different from $G$ and $H$ in order to get a proper cubic $\mathcal{C}(X)$.

3.1 Theorem 1

Let $P$ be a point on $\mathcal{C}(X)$ such that the perspector of $ABC$ and $X_aX_bX_c$ is $Q$. Then $Q$ is the second intersection of the line $XP$ with the rectangular circum-hyperbola passing through $P$ (and through the orthocenters $H_a, H_b, H_c$ of triangles $PBC, PCA, PAB$ respectively). $Q'$ is defined likewise with $X'$ instead of $X$.

3.2 Theorem 2

Since $\mathcal{C}(k) = \mathcal{C}(1/k)$, the locus of perspectors $Q$ and $Q'$ is the union of two circum-cubics denoted by $\mathcal{D}(k)$ and $\mathcal{D}(1/k)$ or equivalently $\mathcal{D}(X)$ and $\mathcal{D}(X')$. $\mathcal{D}(k)$ passes through $X, H, X_5$, meets the sidelines of $ABC$ at $U' = BC \cap XA'$, $V' = CA \cap XB'$ and $W' = AB \cap XC'$, intersects the three altitudes again at the points $U_a, U_b, U_c$ met in §1.5. $\mathcal{D}(1/k)$ has analogous properties.

For example, $\mathcal{D}(0)$ is the Napoleon cubic $K005 = \mathcal{N}_a$ and $\mathcal{D}(\infty)$ is the cubic we call $K060 = \mathcal{K}_n$ which is a very remarkable circular pivotal isocubic. More informations about $\mathcal{K}_n$ can be found in [1] §4.3.1.

3.3 Theorem 3

All the cubics $\mathcal{D}(k)$ form a pencil of circum-cubics generated by $\mathcal{N}_a = \mathcal{D}(0)$ and $\mathcal{K}_n = \mathcal{D}(\infty)$. Taking

$$\mathcal{N}_a = \sum_{\text{cyclic}} [(b^2 - c^2)^2 - a^2(b^2 + c^2)] x(c^2y^2 - b^2z^2) = 0$$

and

$$\mathcal{K}_n = \sum_{\text{cyclic}} 2S_A x [(4S_B^2 - a^2c^2)y^2 - (4S_C^2 - a^2b^2)z^2] = 0$$

we have $\mathcal{D}(k) = \mathcal{N}_a - k \mathcal{K}_n = 0$.

Remarks:

1. There are only three pivotal isocubics in this pencil: $\mathcal{N}_a = K005$, $\mathcal{K}_n = K060$ and $\mathcal{D}(1) = K049$.

2. $K116 = \mathcal{C}(2) = \mathcal{C}(1/2) = \mathcal{D}(2)$ and this is the only case where $\mathcal{C}(k)$ and $\mathcal{D}(k)$ are identical.

3. The base-points of the pencil are $A, B, C, H, X_5$ and the four $I_x$–anticevian points. See Table 23 in [2].

3.4 Theorem 4

The asymptotic directions of $\mathcal{D}(k)$ are those of $p\mathcal{K}(X6,Y_d)$ where the point $Y_d$ is defined by

$$\overrightarrow{OY_d} = \frac{1-k}{2} \overrightarrow{OH}.$$
3.5 Other remarkable \( D(k) \)

3.5.1 The cubic \( K122 = D(-1) \)

\( D(k) = D(1/k) \) if and only if \( k = -1 \). \( D(-1) \) passes through \( X_4, X_5, X_{20}, X_{485}, X_{486} \) (Vecten points) and its asymptotes are parallel to those of the orthicubic.

3.5.2 The cubics \( K049 = D(1), K127 = D(3) \) and \( K123 = D(-1/2) \)

\( D(k) \) has three concurring asymptotes if and only if \( k = 1 \) or \( k = 3 \) or \( k = -1/2 \).

- When \( k = 1 \), \( C(k) \) is the union of the altitudes and we do not obtain a proper \( D(k) \). Nevertheless, the member of the pencil as seen in §3.3 is an interesting cubic with three concurring (at \( X_{51} \)) asymptotes parallel to those of the McCay cubic. This cubic \( D(1) \) is in fact the McCay cubic of the orthic triangle.

- When \( k = 3 \), \( X \) is the reflection of \( L = X_{20} \) about \( H \). The asymptotes of \( D(3) \) are parallel to the altitudes and concur at \( E_{280} \), reflection of \( X_5 \) in \( H \).

- When \( k = -1/2 \), \( X \) is the reflection of \( X_5 \) in \( H \). The asymptotes of \( D(-1/2) \) concur at \( G \).

4 More about \( H \) and \( C(k) \)

We suppose again that \( C(k) \) is a proper cubic curve i.e. \( k \neq 1 \) and \( k \neq 1/3 \).

4.1 Tangent and normal at \( H \) to all \( C(k) \)

The equation of the tangent \( T_H \) is:

\[
\sum_{\text{cyclic}} (b^2 - c^2) S_A^2 [a^2 (b^2 + c^2) - (b^2 - c^2)^2] x = 0
\]

This is the line through \( H \) and \( X_{54} \) (isogonal conjugate of the nine point center \( X_5 \)). The normal at \( H \) is obviously perpendicular at \( H \) to \( T_H \).

4.2 Polar conic of \( H \) in all \( C(k) \)

The polar conic of \( H \) in \( C(k) \) is the rectangular hyperbola \( \mathcal{H} \) which passes through \( H, L, X_{393} \) and the three harmonic conjugates of \( H \) with respect to \( A \) and \( A' \), \( B \) and \( B' \), \( C \) and \( C' \). It is obviously tangent at \( H \) to \( T_H \). Its equation is:

\[
\sum_{\text{cyclic}} (b^2 - c^2) S_A (S_A x^2 - a^2 yz) = 0
\]

Its center is \( \Omega_H \) on the line \( X_3X_{125} \) (\( X_{125} \) is the center of the Jerabek hyperbola). \( \mathcal{H} \) is homothetic to the rectangular circum-hyperbola with center \( X_{136} \).

4.3 Osculating circle at \( H \) to all \( C(k) \)

The curvature of \( C(k) \) at \( H \) is double of the curvature of \( \mathcal{H} \) at the same point (theorem of Moutard). It is therefore easy to construct the center of curvature \( R \) and the osculating circle \( \gamma_H \) at \( H \):

- reflect \( \Omega_H \) about \( H \) to get the point \( E \),
– the normal at $H$ to $C(k)$ intersects the perpendicular at $E$ to $HE$ at the point $R'$ (center of curvature of the rectangular hyperbola at $H$),
– the center of curvature $R$ is the midpoint of $HR'$ and the osculating circle $\gamma_H$ is that with diameter $HR'$.

4.4 Coresiduals

• The coresidual $R_1$ of $A, B, C, H$ lies on the rectangular hyperbola $H_1$ through $A', B', C', H, X_{382}, X_{399}$ which is tangent at $H$ to $T_H$. $R_1$ is the intersection of the lines $A'U, B'V, C'W$ (See §1.5). An equation of $H_1$ is:

$$
\sum_{\text{cyclic}} (b^2 - c^2) \left[ 4S_A^2 x^2 - \left( (b^2 - c^2)^2 + a^2(b^2 + c^2 - 2a^2) \right) yz \right] = 0
$$

• The coresidual $R_2$ of $A', B', C', H$ lies on the rectangular hyperbola $H_2$ centered at $X_{137}$, passing through $A, B, C, H, X$ which is also tangent at $H$ to $T_H$. An equation of $H_2$ is:

$$
\sum_{\text{cyclic}} (b^2 - c^2) \left[ (b^2 - c^2)^2 - a^2(b^2 + c^2) \right] yz = 0
$$

Remark: these two points $R_1$ and $R_2$ are collinear with $H$. Hence, $R_2$ is the second intersection of the line $HR_1$ with $H_2$.

5 The pencil $\mathcal{F}(k)$ of cubics generated by $C(k)$ and $D(k)$

For any $k$ such that $k \neq 2$ and $k \neq 1/3$, the cubics $C(k)$ and $D(k)$ are distinct and generate a pencil of cubics $\mathcal{F}(k)$ passing through $A, B, C, H, X$ and four other points $P_i, i \in \{1, 2, 3, 4\}$ which depend on $X$. These four points lie on the Euler-Morley quintic $Q003$. Indeed, the elimination of $k$ between the equations of $C(k)$ and $D(k)$ gives the Euler line and $Q003$.

If $t$ is a real number or $\infty$, we shall write any cubic of this pencil under the form $\mathcal{F}(k, t) = (1 - t) C(k) + t D(k)$ so that $\mathcal{F}(k, 0) = C(k)$ and $\mathcal{F}(k, 1) = D(k)$.

In this pencil we find several remarkable cubics obtained with certain specific values of $t$ which we examine in the next paragraphs. Note that these cubics are not necessarily all distinct.

5.1 $\mathcal{F}(k)$ contains 3 $pK$s

There are three pivotal in $\mathcal{F}(k)$ obtained when

• $t = \infty$ giving $pK(X_6, X)$, a cubic of the Euler pencil.

• $t = \frac{k - 1}{k - 2}$ giving $pK(X_4 \times X, X_4)$, a cubic of the pencil generated by the Orthocubic $K006$ and the McCay orthic cubic $K049$.

• $t = \frac{(k - 1)^2}{k(k - 2)}$ giving $pK(X_4 \times P, P)$ with $P$ on the Jerabek hyperbola, a cubic of the pencil generated by the McCay cubic $K003$ and the Lucas cubic $K007$. 

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5.2 $\mathcal{F}(k)$ contains 3 $\mathcal{K}^+$ and one of them is a stelloid

Recall that $\mathcal{K}^+$ denotes a cubic with concurring asymptotes which becomes $\mathcal{K}^{++}$ when the point of concurrence lies on the cubic.

There are three $\mathcal{K}^+$ in $\mathcal{F}(k)$ obtained when

- $t = \frac{k^2 - 3k + 1}{(k - 2)k}$ giving a stelloid with asymptotes parallel to those of the McCay cubic $\mathcal{K}003$. These asymptotes concur at $N$ defined by $\overrightarrow{GN} = \frac{2k^2}{3k - 1} \overrightarrow{GX_{51}}$.

- $t = \frac{k^2 + 1}{k(k - 2)}$, giving a $\mathcal{K}^+$ with asymptotes concurring at $X_2$. These asymptotes are parallel to those of $p\mathcal{K}(X_6, T)$ where $T$ is the point on the Euler line defined by $\overrightarrow{OT} = 3k^2 \overrightarrow{OH}$.

- $t = \frac{(k - 1)^2}{(k - 2)(k + 1)}$ giving a $\mathcal{K}^+$ with asymptotes perpendicular to the sidelines of $ABC$ and concurring at $S$ on the Euler line defined by $\overrightarrow{OS} = \frac{k(k + 1)}{3k - 1} \overrightarrow{OH}$.

5.3 $\mathcal{F}(k)$ contains one circular cubic

This cubic is obtained with $t = \frac{k}{k - 2}$ and this is the orthopivotal cubic $\mathcal{O}(X)$. See [3] for informations.

Any such cubic contains nine fixed points namely $A, B, C, X_4, X_{13}, X_{14}, X_{30}$ and the circular points at infinity. Hence all these cubics are in a same pencil generated by the Neuberg cubic $\mathcal{K}001$ and the 7th Brocard cubic $\mathcal{K}023$.

The singular focus $F$ lies on the line passing through $X_2, X_{98}, X_{110}$, etc. It is defined by $\overrightarrow{GF} = -\frac{1}{3k - 1} \overrightarrow{GX_{110}}$.

5.4 Two remarkable examples

**Example 1** : With $k = -1$, we find the cubics of the pencil generated by the Darboux cubic $\mathcal{K}004$ (a $\mathcal{K}^+$) and the Lucas cubic $\mathcal{K}007$ which are two $p\mathcal{K}$s. The third one is $\mathcal{K}329$.

The remaining base-points of the pencil $\mathcal{F}(-1)$ are the CPCC points. See definition and properties at Table 11 in [2].

$\mathcal{C}(-1)$ is the Soddy cubic $\mathcal{K}032$ and $\mathcal{D}(-1)$ is $\mathcal{K}122$.

The circular cubic is $\mathcal{K}313$ and the stelloid is $\mathcal{K}268$.

The third $\mathcal{K}^+$ is unlisted in [2].

**Example 2** : With $k = 1/2$, we find the cubics of the pencil generated by the Napoleon cubic $\mathcal{K}005$ and the McCay orthic cubic $\mathcal{K}049$ (a stelloid) which are two $p\mathcal{K}$s. The third one is $\mathcal{K}060$, a circular cubic.

The remaining base-points of the pencil $\mathcal{F}(1/2)$ are the $I_x$—anticevian points points. See definition and properties at Table 23 in [2].

$\mathcal{C}(1/2)$ is $\mathcal{K}116$ and $\mathcal{D}(1/2)$ is $\mathcal{K}125$.

The two other $\mathcal{K}^+$ are $\mathcal{K}123$ and $\mathcal{K}127$. 7
References


