On two Remarkable Pencils of Cubics of the Triangle Plane

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Abstract

Let X be a center[1] in the plane of the reference triangle ABC. For any point P, denote by X_a, X_b, X_c this same center in the triangles PBC, PCA, PAB respectively. We seek the locus $\mathcal{E}(X)$ of point P such that the triangles ABC and $X_aX_bX_c$ are perspective. The general case is a difficult problem and, in this paper, we only study the particular situation when X is a center on the Euler line. We shall meet several interesting cubics.

1 The cubics $\mathcal{C}(k)$

Let k be a real number or ∞. Denote by X the point such that $\overrightarrow{OX} = k\ \overrightarrow{OH}$ (where O = circumcenter, H = orthocenter) and denote by X' the point on the Euler line defined by $\overrightarrow{OX'} = 1/k\ \overrightarrow{OH}$. X' is clearly the harmonic conjugate of X with respect to H and $L = X_{20}$ (de Longchamps point).

1.1 A trivial case

When $X = G$ (centroid), the locus $\mathcal{E}(G)$ is the entire plane. More precisely, for any P, the lines joining the vertices of ABC to the centroids of the triangles PBC, PCA, PAB concur at the complement of the complement of P. Hence, from now on, we take $X \neq G$ i.e. $k \neq 1/3$.

1.2 Theorem 1

For any center X which is not G, the locus $\mathcal{E}(X)$ is the union of the line at infinity, the circumcircle[2] and a cubic curve denoted by $\mathcal{C}(X)$.

Moreover, $\mathcal{C}(X) = \mathcal{C}(X')$ or equivalently $\mathcal{C}(k) = \mathcal{C}(1/k)$ (when $k = 3$, see §2.1 below). $\mathcal{C}(X)$ and $\mathcal{C}(X')$ pass through A, B, C, the reflections A', B', C' of A, B, C about the sidelines, H, X and X'.

1.3 Theorem 2

All the cubics $\mathcal{C}(X)$ obtained are in the same pencil of cubics which is generated by the Neuberg cubic K001 (obtained for $X = O$ or $X = X_{30}$ i.e. $k = 0$ or $k = \infty$) and the union of the three altitudes ($X = H$ i.e. $k = 1$). Hence all these cubics pass through A, B, C, their reflections A', B', C' in the sidelines of ABC and H. H being a triple point on the second (degenerated) cubic, all the cubics share the same tangent, the same polar conic (which is a rectangular hyperbola) and the same osculating circle at H (see §4 for more details).

[1] See [4], p.46

[2] When P lies on the circumcircle, the triangles ABC and $X_aX_bX_c$ are homothetic at Q which therefore also lies on a circle.

If we write an equation of the Neuberg cubic under the form

$$\mathcal{N} = \sum_{\text{cyclic}} (a^2 S_A - 2S_B S_C)\, x\, (c^2 y^2 - b^2 z^2) = 0$$

and an equation of the union of the three altitudes as

$$\mathcal{A} = (S_A x - S_B y)(S_B y - S_C z)(S_C z - S_A x) = 0,$$

a computation gives an equation of $\mathcal{C}(k)$ which rewrites as :

$$(k-1)^2 \mathcal{N} - 4\,k\mathcal{A} = 0$$

1.4 Theorem 3

• If $k^2 - 3k + 1 \neq 0$, the locus of point M whose polar conic (with respect to $\mathcal{C}(k)$) is a rectangular hyperbola is the Euler line. From this, there are three points (not always distinct when $k = \pm 1$) on $\mathcal{C}(k)$ in this situation : H, X and X'.

• If $k^2 - 3k + 1 = 0 \iff k = (3 \pm \sqrt{5})/2$, all the points in the plane have a polar conic which is a rectangular hyperbola. The corresponding cubic[3] $\mathcal{C}(k)$ is called the 'Golden Cubic" and is studied in §2.3.

1.5 Theorem 4

$\mathcal{C}(X)$ meets the sidelines of ABC again at three points U, V, W which can be constructed as follows : if G_a is the projection of G on the altitude AH and if U_a is the barycentre of the system $\{(A, k), (G_a, -3)\}$, then the line XU_a meets BC at U.

Remark : when $k = 3$, the barycentre is the point at infinity of the altitude AH and X is the reflection L_1 of L about H : in this case, UVW is the pedal triangle of L_1 (see §2.1).

1.6 Theorem 5

The asymptotic directions of $\mathcal{C}(k)$ are those of $p\mathcal{K}(X6, Y_c)$ where the point Y_c is defined by

$$\overrightarrow{OY_c} = \frac{k^2 - 3k + 1}{k}\,\overrightarrow{OH}.$$

2 Some examples

2.1 The cubic K117 $= \mathcal{C}(3)$

When $k = 3$ i.e. X is the reflection L_1 of L about H [4], we obtain the cubic $\mathcal{C}(3)$ passing through G, tangent to GK at G, with a very simple equation :

$$\sum_{\text{cyclic}} x\, \left[(3a^2 - 3b^2 + c^2)y^2 - (3a^2 + b^2 - 3c^2)z^2\right] = 0$$

The homothety with center G, ratio 4 transforms A, B, C into three points A_4, B_4, C_4 lying on the curve. The asymptotes of $\mathcal{C}(3)$ are parallel to those of the Thomson cubic.

Remember also (see §1.5) that $\mathcal{C}(3)$ passes through the vertices U, V, W of the pedal triangle of L_1. The homothetic of K (center G, ratio 10) also lie on the curve : it is the coresidual of A, B, C, H and is collinear with U and A', V and B', W and C'.

[3] Observe here that we have two inverse values and therefore one single cubic.

[4] This point has first barycentric coordinate $3(b^2 - c^2)^2 + a^2(2b^2 + 2c^2 - 5a^2)$ and is not mentioned in [5].

2.2 The Soddy cubic K032 $= \mathcal{C}(-1)$

When we take $k = -1$ or equivalently $X = X' = L$, we get a very interesting case $\mathcal{C}(L)$ of such cubic and we shall call it the Soddy cubic since it passes through the eight Soddy centers. It is tangent at L to the Euler line and meets the sidelines of ABC on the lines passing through L and the midpoints of the altitudes.

2.3 The Golden Cubic K115

In §1.4, we had found $k = (3 \pm \sqrt{5})/2$. We associate the points Φ and Φ' (on the Euler line) to the real numbers $(3+\sqrt{5})/2$ and $(3-\sqrt{5})/2$ respectively. Hence,

$$\overrightarrow{O\Phi} = \frac{3+\sqrt{5}}{2}\,\overrightarrow{OH} \quad \text{and} \quad \overrightarrow{O\Phi'} = \frac{3-\sqrt{5}}{2}\,\overrightarrow{OH}$$

which rewrites under the form

$$\overrightarrow{H\Phi} = \frac{1+\sqrt{5}}{2}\,\overrightarrow{OH} \quad \text{and} \quad \overrightarrow{H\Phi'} = \frac{1-\sqrt{5}}{2}\,\overrightarrow{OH}$$

We call $\mathcal{C}(\Phi) = \mathcal{C}(\Phi')$ the Golden Cubic which obviously passes through $A, B, C, A', B', C', H, \Phi$ and Φ'. Its barycentric equation is :

$$\sum_{\text{cyclic}} y\,z\,[y\,f(a,b,c) - z\,f(a,c,b)] = 0$$

where $f(a,b,c) = (b^2-c^2)^3 + a^2\left[b^2(c^2+a^2-2b^2)+c^4\right]$.

Since all the polar conics are rectangular hyperbolas, $\mathcal{C}(\Phi)$ has three real concurring asymptotes making 60° angles with one another. They are perpendicular to the sidelines of the Morley triangle[5] and concur at the point Z intersection of the parallel at G to the Brocard line OK and the line KX_{22}[6], but this point does not lie on the curve. This type of cubic is called a $\mathcal{K}_{60}^{+}$ in [1]. The first barycentric coordinate of Z is $a^2(b^2c^2+c^2a^2+a^2b^2-b^4-c^4)$ and this point is not mentioned in [5].

2.4 Other remarkable $\mathcal{C}(k)$

2.4.1 The cubic K116 $= \mathcal{C}(2) = \mathcal{C}(1/2)$

$\mathcal{C}(2)$ passes through X_4, X_5, X_{382}. See remark in §3.3 below.

2.4.2 The cubic K119 $= \mathcal{C}(\Psi) = \mathcal{C}(\Psi')$

With $k = (7+\sqrt{33})/4$ and $k = (7-\sqrt{33})/4$, we obtain the points Ψ and Ψ' on the Euler line. $\mathcal{C}(\Psi)$ passes through $X_4, X_{17}, X_{18}, \Psi$ and Ψ'. Its asymptotes are parallel to those of the Napoleon cubic.

2.4.3 The cubic K120 $= \mathcal{C}(\Theta) = \mathcal{C}(\Theta')$

With $k = 2+\sqrt{3}$ and $k = 2-\sqrt{3}$, we obtain the points Θ and Θ' on the Euler line. $\mathcal{C}(\Theta)$ passes through X_4, Θ and Θ'. Its asymptotes are parallel to those of the orthocubic. The points Θ and Θ' are related to the isodynamic points X_{15}, X_{16} and the Vecten points X_{485}, X_{486} since we have the four following collinearities :

$$\Theta, X_{15}, X_{485} - \Theta, X_{16}, X_{486} - \Theta', X_{15}, X_{486} - \Theta', X_{16}, X_{485}$$

The circle with diameter $\Theta\Theta'$ is centered at X_{382} and its radius is $\sqrt{3}\,OH$.

[5] i.e. parallel to those of the McCay cubic.

[6] X_{22} is the Exeter point.

3 Locus of the perspectors : the cubics $\mathcal{D}(k)$

In this paragraph we suppose X different from G and H in order to get a proper cubic $\mathcal{C}(X)$.

3.1 Theorem 1

Let P be a point on $\mathcal{C}(X)$ such that the perspector of ABC and $X_aX_bX_c$ is Q. Then Q is the second intersection of the line XP with the rectangular circum-hyperbola passing through P (and through the orthocenters H_a, H_b, H_c of triangles PBC, PCA, PAB respectively). Q' is defined likewise with X' instead of X.

3.2 Theorem 2

Since $\mathcal{C}(k) = \mathcal{C}(1/k)$, the locus of perspectors Q and Q' is the union of two circum-cubics denoted by $\mathcal{D}(k)$ and $\mathcal{D}(1/k)$ or equivalently $\mathcal{D}(X)$ and $\mathcal{D}(X')$. $\mathcal{D}(k)$ passes through X, H, X_5, meets the sidelines of ABC at $U' = BC \cap XA'$, $V' = CA \cap XB'$ and $W' = AB \cap XC'$, intersects the three altitudes again at the points U_a, U_b, U_c met in §1.5. $\mathcal{D}(1/k)$ has analogous properties.

For example, $\mathcal{D}(0)$ is the Napoleon cubic K005 $= \mathcal{N}_a$ and $\mathcal{D}(\infty)$ is the cubic we call K060 $= \mathcal{K}_n$ which is a very remarkable circular pivotal isocubic. More informations about $\mathcal{K}_n$ can be found in [1] §4.3.1.

3.3 Theorem 3

All the cubics $\mathcal{D}(k)$ form a pencil of circum-cubics generated by $\mathcal{N}_a = \mathcal{D}(0)$ and $\mathcal{K}_n = \mathcal{D}(\infty)$. Taking

$$\mathcal{N}_a = \sum_{\text{cyclic}} \left[(b^2-c^2)^2 - a^2(b^2+c^2)\right] x(c^2y^2 - b^2z^2) = 0$$

and

$$\mathcal{K}_n = \sum_{\text{cyclic}} 2S_A\, x \left[(4S_B^2 - a^2c^2)y^2 - (4S_C^2 - a^2b^2)z^2\right] = 0$$

we have $\mathcal{D}(k) = \mathcal{N}_a - k\,\mathcal{K}_n = 0$.

Remarks :

1. There are only three pivotal isocubics in this pencil : $\mathcal{N}_a$ = K005, $\mathcal{K}_n$ = K060 and $\mathcal{D}(1)$ = K049.
2. K116 $= \mathcal{C}(2) = \mathcal{C}(1/2) = \mathcal{D}(2)$ and this is the only case where $\mathcal{C}(k)$ and $\mathcal{D}(k)$ are identical.
3. The base-points of the pencil are A, B, C, H, X_5 and the four I_x-anticevian points. See Table 23 in [2].

3.4 Theorem 4

The asymptotic directions of $\mathcal{D}(k)$ are those of $p\mathcal{K}(X6, Y_d)$ where the point Y_d is defined by

$$\overrightarrow{OY_d} = \frac{1-k}{2}\,\overrightarrow{OH}.$$

3.5 Other remarkable $\mathcal{D}(k)$

3.5.1 The cubic K122 $= \mathcal{D}(-1)$

$\mathcal{D}(k) = \mathcal{D}(1/k)$ if and only if $k = -1$. $\mathcal{D}(-1)$ passes through $X_4, X_5, X_{20}, X_{485}, X_{486}$ (Vecten points) and its asymptotes are parallel to those of the orthocubic.

3.5.2 The cubics K049 $= \mathcal{D}(1)$, K127 $= \mathcal{D}(3)$ and K123 $= \mathcal{D}(-1/2)$

$\mathcal{D}(k)$ has three concurring asymptotes if and only if $k = 1$ or $k = 3$ or $k = -1/2$.

- When $k = 1$, $\mathcal{C}(k)$ is the union of the altitudes and we do not obtain a proper $\mathcal{D}(k)$. Nevertheless, the member of the pencil as seen in §3.3 is an interesting cubic with three concurring (at X_{51}) asymptotes parallel to those of the McCay cubic. This cubic $\mathcal{D}(1)$ is in fact the McCay cubic of the orthic triangle.

- When $k = 3$, X is the reflection of $L = X_{20}$ about H. The asymptotes of $\mathcal{D}(3)$ are parallel to the altitudes and concur at E_{280}, reflection of X_5 in H.

- When $k = -1/2$, X is the reflection of X_5 in H. The asymptotes of $\mathcal{D}(-1/2)$ concur at G.

4 More about H and $\mathcal{C}(k)$

We suppose again that $\mathcal{C}(k)$ is a proper cubic curve i.e. $k \neq 1$ and $k \neq 1/3$.

4.1 Tangent and normal at H to all $\mathcal{C}(k)$

The equation of the tangent $\mathcal{T}_H$ is :

$$\sum_{\text{cyclic}} (b^2 - c^2) S_A^2 [a^2(b^2 + c^2) - (b^2 - c^2)^2]\, x = 0$$

This is the line through H and X_{54} (isogonal conjugate of the nine point center X_5). The normal at H is obviously perpendicular at H to $\mathcal{T}_H$.

4.2 Polar conic of H in all $\mathcal{C}(k)$

The polar conic of H in $\mathcal{C}(k)$ is the rectangular hyperbola $\mathcal{H}$ which passes through H, L, X_{393} and the three harmonic conjugates of H with respect to A and A', B and B', C and C'. It is obviously tangent at H to $\mathcal{T}_H$. Its equation is :

$$\sum_{\text{cyclic}} (b^2 - c^2) S_A (S_A x^2 - a^2 yz) = 0$$

Its center is Ω_H on the line X_3X_{125} (X_{125} is the center of the Jerabek hyperbola). $\mathcal{H}$ is homothetic to the rectangular circum-hyperbola with center X_{136}.

4.3 Osculating circle at H to all $\mathcal{C}(k)$

The curvature of $\mathcal{C}(k)$ at H is double of the curvature of $\mathcal{H}$ at the same point (theorem of Moutard). It is therefore easy to construct the center of curvature R and the osculating circle γ_H at H :
– reflect Ω_H about H to get the point E,

– the normal at H to $\mathcal{C}(k)$ intersects the perpendicular at E to HE at the point R' (center of curvature of the rectangular hyperbola at H),

– the center of curvature R is the midpoint of HR' and the osculating circle γ_H is that with diameter HR'.

4.4 Coresiduals

- The coresidual R_1 of A, B, C, H lies on the rectangular hyperbola $\mathcal{H}_1$ through $A', B', C', H, X_{382}, X_{399}$ which is tangent at H to $\mathcal{T}_H$. R_1 is the intersection of the lines $A'U, B'V, C'W$ (See §1.5). An equation of $\mathcal{H}_1$ is :

$$\sum_{\text{cyclic}} (b^2 - c^2) \left[4S_A^2 x^2 - \left((b^2 - c^2)^2 + a^2(b^2 + c^2 - 2a^2)\right) yz\right] = 0$$

- The coresidual R_2 of A', B', C', H lies on the rectangular hyperbola $\mathcal{H}_2$ centered at X_{137}, passing through A, B, C, H, X_5 which is also tangent at H to $\mathcal{T}_H$. An equation of $\mathcal{H}_2$ is :

$$\sum_{\text{cyclic}} (b^2 - c^2) \left[(b^2 - c^2)^2 - a^2(b^2 + c^2)\right] yz = 0$$

Remark : these two points R_1 and R_2 are collinear with H. Hence, R_2 is the second intersection of the line HR_1 with $\mathcal{H}_2$.

5 The pencil $\mathcal{F}(k)$ of cubics generated by $\mathcal{C}(k)$ and $\mathcal{D}(k)$

For any k such that $k \neq 2$ and $k \neq 1/3$, the cubics $\mathcal{C}(k)$ and $\mathcal{D}(k)$ are distinct and generate a pencil of cubics $\mathcal{F}(k)$ passing through A, B, C, H, X and four other points $P_i, i \in \{1, 2, 3, 4\}$ which depend on X. These four points lie on the Euler-Morley quintic Q003. Indeed, the elimination of k between the equations of $\mathcal{C}(k)$ and $\mathcal{D}(k)$ gives the Euler line and Q003.

If t is a real number or ∞, we shall write any cubic of this pencil under the form $\mathcal{F}(k, t) = (1 - t)\, \mathcal{C}(k) + t\, \mathcal{D}(k)$ so that $\mathcal{F}(k, 0) = \mathcal{C}(k)$ and $\mathcal{F}(k, 1) = \mathcal{D}(k)$.

In this pencil we find several remarkable cubics obtained with certain specific values of t which we examine in the next paragraphs. Note that these cubics are not necessarily all distinct.

5.1 $\mathcal{F}(k)$ contains 3 $p\mathcal{K}$s

There are three pivotal in $\mathcal{F}(k)$ obtained when

- $t = \infty$ giving $p\mathcal{K}(X_6, X)$, a cubic of the Euler pencil.
- $t = \dfrac{k-1}{k-2}$ giving $p\mathcal{K}(X_4 \times X, X_4)$, a cubic of the pencil generated by the Orthocubic K006 and the McCay orthic cubic K049.
- $t = \dfrac{(k-1)^2}{k(k-2)}$ giving $p\mathcal{K}(X_4 \times P, P)$ with P on the Jerabek hyperbola, a cubic of the pencil generated by the McCay cubic K003 and the Lucas cubic K007.

5.2 $\mathcal{F}(k)$ contains 3 $\mathcal{K}^+$ and one of them is a stelloid

Recall that $\mathcal{K}^+$ denotes a cubic with concurring asymptotes which becomes $\mathcal{K}^{++}$ when the point of concurrence lies on the cubic.

There are three $\mathcal{K}^+$ in $\mathcal{F}(k)$ obtained when

- $t = \dfrac{k^2 - 3k + 1}{(k-2)k}$ giving a stelloid with asymptotes parallel to those of the McCay cubic K003. These asymptotes concur at N defined by $\overrightarrow{GN} = \dfrac{2k^2}{3k-1}\,\overrightarrow{GX_{51}}$.
- $t = \dfrac{k^2+1}{k(k-2)}$, giving a $\mathcal{K}^+$ with asymptotes concurring at X_2. These asymptotes are parallel to those of $p\mathcal{K}(X_6, T)$ where T is the point on the Euler line defined by $\overrightarrow{OT} = 3k^2\,\overrightarrow{OH}$.
- $t = \dfrac{(k-1)^2}{(k-2)(k+1)}$ giving a $\mathcal{K}^+$ with asymptotes perpendicular to the sidelines of ABC and concurring at S on the Euler line defined by $\overrightarrow{OS} = \dfrac{k(k+1)}{3k-1}\,\overrightarrow{OH}$.

5.3 $\mathcal{F}(k)$ contains one circular cubic

This cubic is obtained with $t = \dfrac{k}{k-2}$ and this is the orthopivotal cubic $\mathcal{O}(X)$. See [3] for informations.

Any such cubic contains nine fixed points namely A, B, C, X_4, X_{13}, X_{14}, X_{30} and the circular points at infinity. Hence all these cubics are in a same pencil generated by the Neuberg cubic K001 and the 7th Brocard cubic K023.

The singular focus F lies on the line passing through X_2, X_{98}, X_{110}, etc. It is defined by $\overrightarrow{GF} = -\dfrac{1}{3k-1}\,\overrightarrow{GX_{110}}$.

5.4 Two remarkable examples

Example 1 : With $k = -1$, we find the cubics of the pencil generated by the Darboux cubic K004 (a $\mathcal{K}^+$) and the Lucas cubic K007 which are two $p\mathcal{K}$s. The third one is K329.

The remaining base-points of the pencil $\mathcal{F}(-1)$ are the CPCC points. See definition and properties at Table 11 in [2].

$\mathcal{C}(-1)$ is the Soddy cubic K032 and $\mathcal{D}(-1)$ is K122.

The circular cubic is K313 and the stelloid is K268.

The third $\mathcal{K}^+$ is unlisted in [2].

Example 2 : With $k = 1/2$, we find the cubics of the pencil generated by the Napoleon cubic K005 and the McCay orthic cubic K049 (a stelloid) which are two $p\mathcal{K}$s. The third one is K060, a circular cubic.

The remaining base-points of the pencil $\mathcal{F}(1/2)$ are the I_x−anticevian points points. See definition and properties at Table 23 in [2].

$\mathcal{C}(1/2)$ is K116 and $\mathcal{D}(1/2)$ is K125.

The two other $\mathcal{K}^+$ are K123 and K127.

References

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