A Generalized Morley Configuration

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Abstract

We generalize some properties of the previous paper "A Morley Configuration".

1 Introduction

Let $M_aM_bM_c$ be the Morley triangle with sidelines denoted L_a, L_b, L_c whose infinite points are I_a, I_b, I_c respectively. These latter points are also the infinite points of the cubic K024, an isogonal stelloidal $n\mathcal{K}$.

In the previous paper [3], it has been proven that :

1. the isogonal conjugates G_a, G_b, G_c of I_a, I_b, I_c are the vertices of an equilateral triangle inscribed in the circumcircle $(\mathcal{O})$ called the circum-tangential triangle. Its vertices also lie on K024.

   If I_a, I_b, I_c are replaced with the infinite points of any three directions making 60° angles with one another, their isogonal conjugates are the vertices of an equilateral triangle again.

2. the isotomic conjugates T_a, T_b, T_c of I_a, I_b, I_c are also the vertices of an equilateral triangle inscribed in the Steiner ellipse $(\mathcal{S})$.

   These points I_a, I_b, I_c are the only points for which $T_aT_bT_c$ is equilateral.

These two equilateral triangles $G_aG_bG_c$ and $T_aT_bT_c$ are homothetic at the Steiner point X_{99} and the circumcircle of $T_aT_bT_c$ is centered at X_{76} on the line X_3X_{99} and is tangent at X_{99} to $(\mathcal{O})$. This gives a conic construction of these points. See figure 1.

In this paper, we propose to generalize these properties and wish to investigate the following questions :

– under which other isoconjugations with pole $\Omega = u : v : w$ (not lying on one sideline of ABC), the triangle $P_aP_bP_c$ whose vertices are the isoconjugates of the infinite points I_a, I_b, I_c of the cubic K024 is also equilateral ?

– what can be said when the infinite points I_a, I_b, I_c are those of the altitudes of the Morley triangle or, equivalently, those of the McCay cubic K003, a well known isogonal stelloidal $p\mathcal{K}$?

– what can be said when the infinite points I_a, I_b, I_c are those of three arbitrary lines making 60° angles with one another ?

Most of the results below were obtained through manipulations of symmetric functions of the roots of third degree polynomials.

In the sequel, $(\mathcal{C}_i)$ denotes the circum-conic with perspector the ETC center X_i and $(\mathcal{C}_X)$ denotes the circum-conic with perspector X.

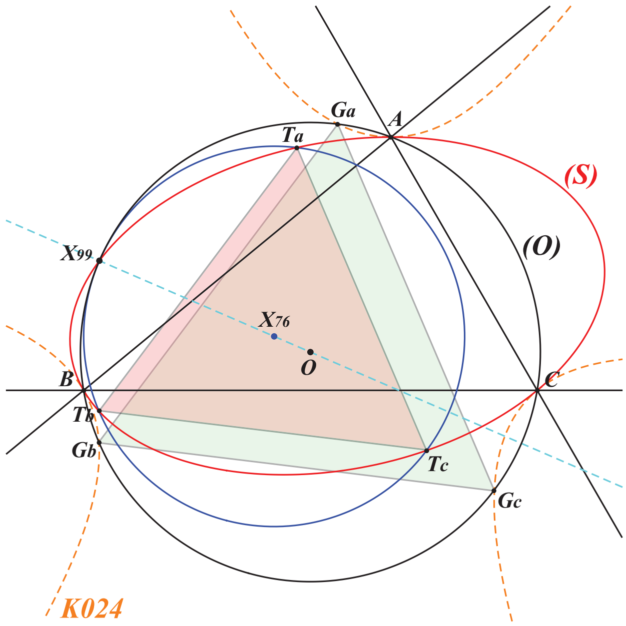

Figure 1: Equilateral triangles $G_aG_bG_c$ and $T_aT_bT_c$

2 The general case

Since K003 and K024 are two isogonal stelloids with same radial center G, it is convenient to consider three lines making 60° angles with one another as a member of a linear pencil generated by the asymptotes A003, A024 of K003, K024 respectively and write these three lines under the form A003 + k A024, k being a real number or ∞.

Recall that A024 (resp. A003) are three lines parallel (resp. perpendicular) to the sidelines of the Morley triangle.

Now, if these lines were obtained from A024 under a rotation with center G and angle θ, it is enough to take

$$k = \frac{4\Delta \cot\theta\,(\cot^2\theta - 3)}{3\cot^2\theta - 1}$$

where Δ is the area of ABC.

For a given pole $\Omega = u : v : w$, the triangle $P_aP_bP_c$ whose vertices are the isoconjugates of the infinite points I_a, I_b, I_c of the three lines is equilateral if and only if three (but actually two are sufficient) conditions are realized.

These conditions on k and on Ω can be written :

$$\mathcal{C}_0 \times \mathcal{C}_1 = 0, \quad \mathcal{C}_0 \times \mathcal{C}_2 = 0, \quad \mathcal{C}_0 \times \mathcal{C}_3 = 0.$$

All of them are linear in k, of global degree 2 in u, v, w for $\mathcal{C}_0$, 3 for the three other conditions. These are examined in the following paragraphs.

2.1 The condition $\mathcal{C}_0$

Let us write $\mathcal{C}_0 = \mathcal{C}'_0 + k\,\mathcal{C}''_0$ where

- $\mathcal{C}'_0 = 0$ if and only if Ω lies on the circum-conic $(\mathcal{C}_{51})$ with perspector X_{51},

- $\mathcal{C}_0'' = 0$ if and only if Ω lies on the circum-conic $(\mathcal{C}_{512})$ with perspector X_{512}.

Note that these two conics meet at four points namely A, B, C (here excluded) and X_{1989}.

This yields the three following situations :

1. when Ω is not on $(\mathcal{C}_{512})$ and then $\mathcal{C}_0'' \neq 0$, one can find only one real number k giving one equilateral triangle $P_aP_bP_c$. In particular, if Ω is on $(\mathcal{C}_{51})$, the sidelines of $P_aP_bP_c$ are parallel to A003.

2. when Ω is on $(\mathcal{C}_{512})$ and then $\mathcal{C}_0'' = 0$,

   - if $\Omega \neq X_{1989}$ we have $\mathcal{C}_0' \neq 0$ hence k must be ∞. Consequently, there is one equilateral triangle $P_aP_bP_c$ whose vertices are the isoconjugates of the infinite points of A024. In particular, with $\Omega = X_6$ we obtain the circum-tangential triangle.
   - if $\Omega = X_{1989}$ we have $\mathcal{C}_0' = 0$ hence k is undetermined therefore there are infinitely many equilateral triangles $P_aP_bP_c$ and all of them are inscribed in the circum-conic $(\mathcal{C}_{1989})$ with perspector X_{1989}.

In conclusion, we see that, for any Ω, it is always possible to find at least one orientation of the three lines that gives one equilateral triangle $P_aP_bP_c$ inscribed in the circum-conic with perspector Ω.

2.2 The conditions $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$

We now suppose that the condition $\mathcal{C}_0$ is not realized and then, the problem has a solution if the conditions $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ are compatible which is generally not the case. Indeed, each condition is realized when Ω lies on a circum-cubic having a node at one vertex of ABC and passing through X_6. The isogonal transform of each circum-cubic is a conic passing through G and a vertex of ABC. The Jacobian of the three conics identically vanishes hence they belong to a same pencil and must have four common points, one of them being G. It follows that at least one of the three remaining common points must be real. The elimination of k between the equations shows that these three points must lie on the circum-conic with perspector X_3 which turns out to be the isogonal transform of the orthic axis (Δ).

This yields the three following situations :

1. if $\Omega = X_6$, the three conditions are realized for any k which we already knew. Hence there are infinitely many equilateral triangles $P_aP_bP_c$ and all of them are inscribed in the circumcircle $(\mathcal{O})$. For $k = 0$, $k = \infty$ we obtain the circum-tangential and circum-normal triangles respectively.

2. if Ω lies on the orthic axis (Δ), the three conditions $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ are identical hence there is one equilateral triangle $P_aP_bP_c$ whose vertices lie on the rectangular circum-hyperbola with perspector Ω. Finally, we find two such triangles counting that obtained with the condition $\mathcal{C}_0 = 0$.

3. if $\Omega \neq X_6$ does not lie on (Δ), there are no other equilateral triangles $P_aP_bP_c$ apart those found in the previous paragraph.

3 Centers of the equilateral triangles

According to the different positions of Ω in the plane and the different orientation of the three lines, the center Q of the equilateral triangle $P_aP_bP_c$ can be determined as follows.

First, recall that for $\Omega = X_6$ we always have $Q = X_3$.

When $\Omega = X_{1989}$, the center Q lies on the perpendicular bisector of OH.

Now, for $\Omega \neq X_6$ and $\Omega \neq X_{1989}$, let us denote by S_1 and S_2 the isogonal conjugates of the infinite points of the trilinear polars of the isoconjugates of Ω under the isoconjugations with poles X_6 and X_{1989} respectively. These two points obviously lie on the circum-circle. Note that S_1 is also the trilinear pole of the line ΩX_6.

In general, Q is the intersection of the perpendicular bisector of S_1S_2 (passing through X_3) and the Steiner line of S_2 (passing through X_4).

When Ω lies on $(\mathcal{C}_{512})$, S_1 and S_2 coincide and when Ω lies on the Fermat line, the construction above is invalid since the perpendicular bisector of S_1S_2 and the Steiner line of S_2 coincide with the Euler line. In this latter case, $S_1 = X_{476}$ and $S_2 = X_{110}$.

The correspondence between Ω on the Fermat line and Q on the Euler line can be established easily since the line ΩQ remains tangent to a fixed hyperbola $(\mathcal{H})$ which is the conic tangent to the Euler line, the Brocard axis, the Fermat line, also the lines X_2X_{99}, X_4X_{3018}, X_5X_{1989}, $X_{20}X_{3163}$, $X_{32}X_{111}$, $X_{112}X_{1113}$, $X_{112}X_{1114}$ and probably more.

The practical construction of Q comes from the Brianchon theorem. For Ω on the Fermat line, successively construct :

- $M = X_2X_{99} \cap X_5X_{1989}$ (this is the singular point of the transformation $\Omega \mapsto Q$),
- $N = X_5X_6 \cap M\Omega$,
- $Q = X_2X_3 \cap X_{574}N$.

Remark : this point M is unlisted in the current edition of ETC with SEARCH = 3.36523271274364.

4 The two special cases $\Omega = X_6$ and $\Omega = X_{1989}$

There are two (and only two) situations where, for any orientation of the three lines, one can always find an equilateral triangle $P_aP_bP_c$.

4.1 $\Omega = X_6$

For completeness, we simply recall that there are infinitely many equilateral triangles inscribed in $(\mathcal{C}_6) = (\mathcal{O})$ and the isogonal conjugates of their vertices are the infinite points of the perpendiculars to their sidelines.

4.2 $\Omega = X_{1989}$

$(\mathcal{C}_{1989})$ is a hyperbola with eccentricity 2 hence having two asymptotes making a 60° angle. The center ω is the complement of X_{1138}, also the $G-$Ceva conjugate of X_{1989}. The focal axis is the line $X_{30}X_{74}$ (the real asymptote of the Neuberg cubic). The real foci are X_{265} and its reflection in ω. The vertices are X_{476} and its reflection X_{5627} in ω. The

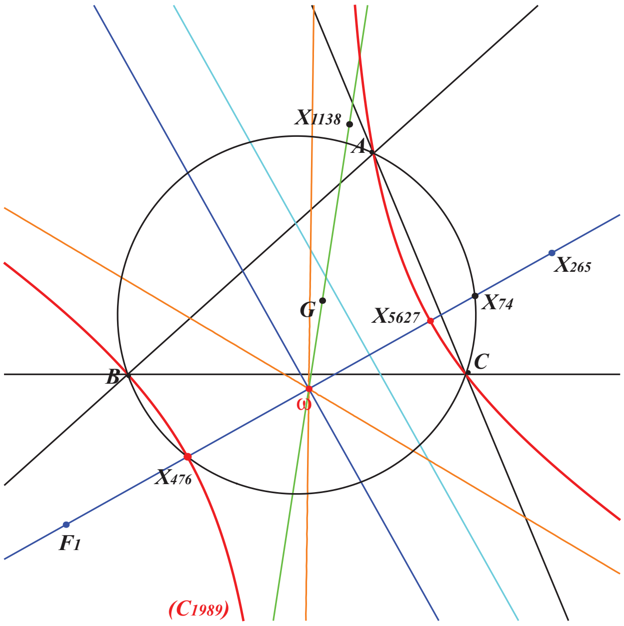

Figure 2: The circum-conic $(\mathcal{C}_{1989})$

directrix corresponding to the focus X_{265} is the perpendicular bisector of OH, the other directrix being obviously its reflection in ω. See figure 2.

As already said, the circum-conics $(\mathcal{C}_{512})$ and $(\mathcal{C}_{51})$ seen above meet again at X_{1989}, the barycentric product of the Fermat points X_{13}, X_{14}.

It follows that the $X_{1989}-$isoconjugates of the infinites points of K003 and K024 are the vertices of two equilateral triangles $\mathcal{T}_{003}$ and $\mathcal{T}_{024}$ inscribed in the circum-conic $(\mathcal{C}_{1989})$ with perspector X_{1989} meeting $(\mathcal{O})$ again at X_{476}.

From the study made in the previous sections, we find the characteristics of the two triangles :

– $\mathcal{T}_{003}$ has center P_{027}, the $X_{1989}-$isoconjugate of X_{110}, and this point lies on K027. The circumcircle of $\mathcal{T}_{003}$ contains X_{265}, X_{476}, X_{1141} and it is orthogonal to $(\mathcal{O})$. The SEARCH number of P_{027} is $-0.4426998453508 96$. P_{027} is now X_{10412} in ETC.

– $\mathcal{T}_{024}$ has center P_{028}, the $X_{1989}-$isoconjugate of X_{74}, and this point lies on K028. The circumcircle of $\mathcal{T}_{024}$ contains X_{265}, X_{476} and it is tangent at X_{476} to $(\mathcal{O})$. The SEARCH number of P_{028} is 0.0941046961049146.

Note that these two points P_{027}, P_{028} lie on the perpendicular bisector of OH and the circle with diameter $P_{027}P_{028}$ also contains X_{265}, X_{476}. This will be generalized below. Hence, these two circumcircles are also orthogonal. See figure 3.

Remark : the three triangles $G_aG_bG_c$, $\mathcal{T}_{003}$, $\mathcal{T}_{024}$ are two by two in perspective at the points X_{265}, X_{476}, X_{1141}. Recall that X_{476} is the trilinear pole of the line X_6X_{1989}.

More generally, it is always possible to draw an equilateral triangle $P_aP_bP_c$ inscribed in $(\mathcal{C}_{1989})$ with given orientation of its sidelines or, equivalently, such that its circumcircle cuts $(\mathcal{O})$ under a given angle. $(\mathcal{C}_{1989})$ is the only circum-conic having this property. This circumcircle always passes through X_{476} and X_{265} hence its center Q lies on the perpendicular bisector of OH.

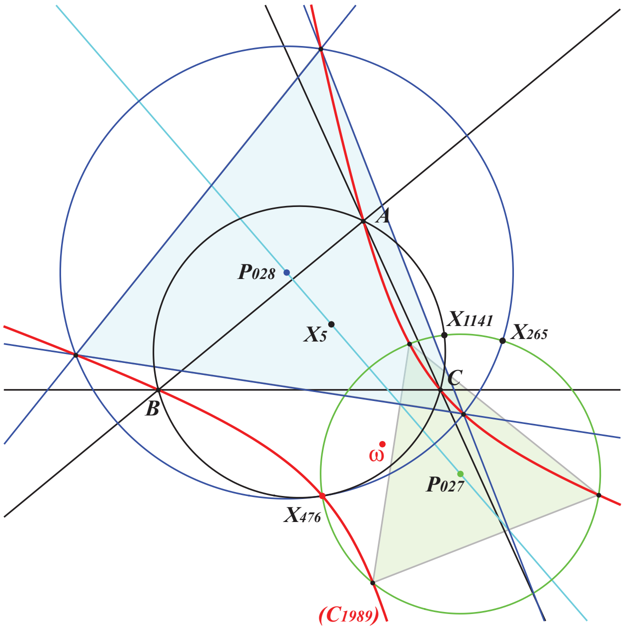

Figure 3: Two equilateral triangles $\mathcal{T}_{003}$ and $\mathcal{T}_{024}$ inscribed in the circum-conic $(\mathcal{C}_{1989})$

The sidelines of $P_aP_bP_c$ are tangent to the parabola with focus X_{265} and directrix the perpendicular bisector of OH.

If the center Q is given, the line passing through X_{265} and the $X_{1989}-$isoconjugate of Q – this latter point lying on $(\mathcal{O})$ – meets $(\mathcal{O})$ again at P_1. The requested circumcircle is that passing through P_1, meeting $(\mathcal{C}_{1989})$ at X_{476} and P_a, P_b, P_c. See figure 4 in which $\mathcal{F}(k)$ is a cubic of the pencil K003, K024.

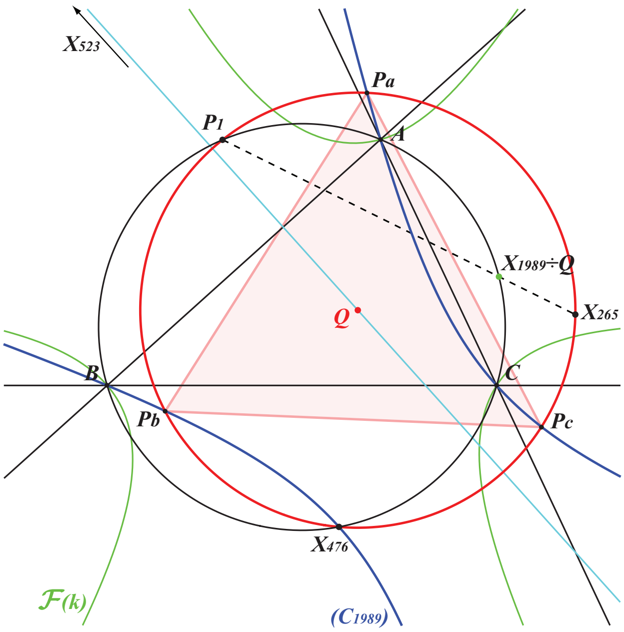

Figure 4: $\Omega = X_{1989}$

Note that the three directions making a 60° angle with one another are those of the sidelines of the triangle $P_aP_bP_c$. In other words, the $X_{1989}-$isoconjugate of P_a is the infinite point of the line P_bP_c.

5 $\Omega \neq X_{1989}$ lies on $(\mathcal{C}_{512})$

The circum-conic $(\mathcal{C}_{512})$ with perspector X_{512} passes through X_2, X_6 (as already mentioned in the introduction) and also many other centers namely X_{25}, X_{37}, X_{42}, X_{111}, X_{251}, X_{263}, X_{308}, X_{393}, X_{493}, X_{494}, X_{588}, X_{589}, X_{694}, X_{941}, X_{967}, X_{1169}, X_{1171}, X_{1218}, X_{1239}, X_{1241}, X_{1383}, X_{1400}, X_{1427}, X_{1880}, X_{1976}, X_{1989}, X_{2054}, X_{2165}, X_{2248}, X_{2350}, X_{2395}, X_{2433}, X_{2963}, X_{2981}, X_{2987}, X_{2998}, X_{3108}, X_{3228}, etc.

Any point Ω on $(\mathcal{C}_{512})$ is the pole of an isoconjugation such that $P_aP_bP_c$ is equilateral and, in this case, the triangle is homothetic to the circum-tangential triangle $G_aG_bG_c$ at S_1, the intersection of $(\mathcal{O})$ with the circum-conic $(\mathcal{C}_\Omega)$ with perspector Ω or, equivalently, the trilinear pole of the line ΩX_6 when $\Omega \neq X_6$. Note that S_1 is the "last" point on the circumcircle and on $(\mathcal{C}_\Omega)$. When $\Omega = X_2$ we find the Steiner point X_{99} again.

The center Q of each equilateral triangle is the intersection of the line OS_1 with the Steiner line of S_1 and consequently, this point Q lies on the third Musselman cubic K028. This gives a generalized conic construction of any such equilateral triangle : it is sufficient to intersect $(\mathcal{C}_\Omega)$ and the circle with center Q passing through S_1, discarding S_1. Recall that this latter circle is tangent at S_1 to $(\mathcal{O})$.

It is interesting to note that the isogonal conjugates P_a^*, P_b^*, P_c^* of P_a, P_b, P_c are three collinear points of the trilinear polar of Ω^* and these points also lie on the sidelines of the circumtangential triangle $G_aG_bG_c$. When Ω traverses $(\mathcal{C}_{512})$, the envelope of this trilinear polar is the Kiepert parabola. See figure 5 where $\Omega = X_{111}$, $S_1 = X_{691}$, Q being unlisted in ETC.

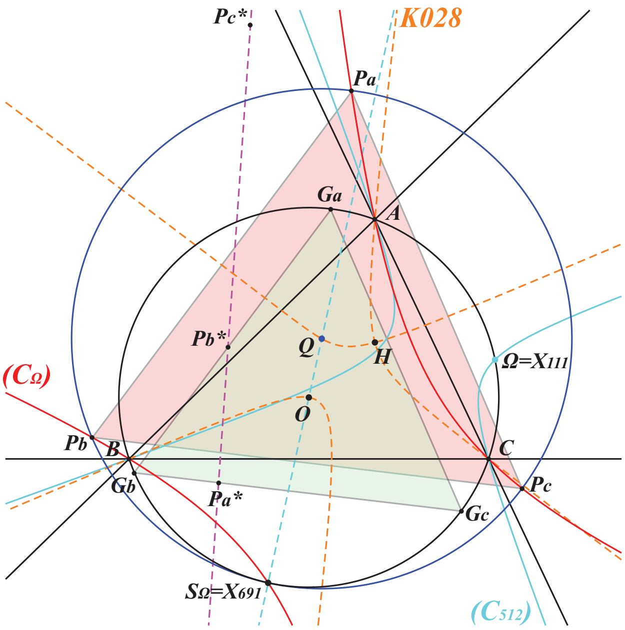

Figure 5: Equilateral triangles $G_aG_bG_c$ and $P_aP_bP_c$ with Ω on $(\mathcal{C}_{512})$

Remark : when Ω is the $X_{512}-$isoconjugate of one of the three infinite points I_a, I_b, I_c

of K024, the triangle $P_aP_bP_c$ is not defined since one of its sidelines is the line at infinity. This corresponds to a point Q at infinity i.e. to the case when the line OS_1 and the Steiner line of S_1 are parallel.

6 $\Omega \neq X_{1989}$ lies on $(\mathcal{C}_{51})$

The isogonal conjugates of the infinite points of K003 are the vertices of the circum-normal triangle $N_aN_bN_c$ which is the reflection of $G_aG_bG_c$ about O.

The circum-conic $(\mathcal{C}_{51})$ with perspector X_{51} passes through X_{112}, X_{1625}, X_{1987}, X_{1989}.

Any point Ω on $(\mathcal{C}_{51})$ is the pole of an isoconjugation such that $P_aP_bP_c$ is equilateral and, in this case, its sidelines are parallel to the asymptotes of K003 i.e. perpendicular to the sidelines of $G_aG_bG_c$ and $N_aN_bN_c$.

X_{51} is the isogonal conjugate of X_{95} hence the barycentric product $M = X_{95} \times \Omega$ lies on $(\mathcal{O})$. The tangent at M to $(\mathcal{O})$ and the Steiner line of M meet at Q which is the center of $P_aP_bP_c$. This point Q lies on the second Musselman cubic K027.

The circumcircle of $P_aP_bP_c$ passes through M and it is orthogonal to $(\mathcal{O})$. It meets $(\mathcal{O})$ again at N which is the trilinear pole of the line ΩX_6 and this point also lies on the conic $(\mathcal{C}_\Omega)$ which is the $\Omega-$isoconjugate of the line at infinity. This latter conic obviously passes through the points P_a, P_b, P_c. See figure 6.

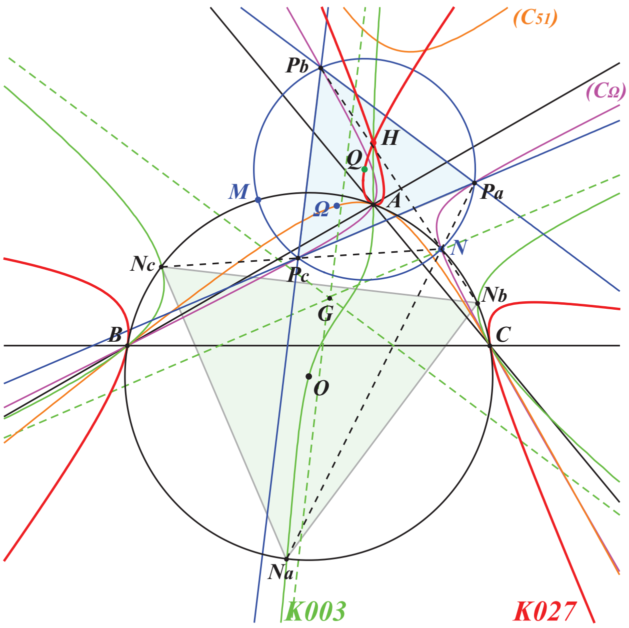

Figure 6: Equilateral triangles $N_aN_bN_c$ and $P_aP_bP_c$ with Ω on $(\mathcal{C}_{51})$

7 Ω lies on the orthic axis (Δ)

When Ω lies on (Δ), the circum-conic $(\mathcal{C}_\Omega)$ is a rectangular hyperbola meeting $(\mathcal{O})$ again at S_1.

It is known that any circle with center on $(\mathcal{C}_\Omega)$ and passing through its antipode on $(\mathcal{C}_\Omega)$ meets $(\mathcal{C}_\Omega)$ again at the vertices of an equilateral triangle. In general, the $\Omega-$isoconjugates of its vertices are not the infinite points of three lines making 60° angles with one another.

This only can occur when the center of the circle is the orthocenter H (resp. S_1) and then this circle passes through S_1 (resp. H). These two equilateral triangles (and their circumcircles) are obviously symmetric with respect to the center of $(\mathcal{C}_\Omega)$, the midpoint of HS_1.

7.1 Ω in $(\Delta) \cap$ K162

(Δ) and K162 meet at three points and one of them is always real. Each point is the pole of an isoconjugation such that $P_aP_bP_c$ is equilateral. In this case, the center of the triangle is one of the intersections (other than A, B, C) of the Orthocubic K006 (and also K028) with the circumcircle $(\mathcal{O})$. See figure 7.

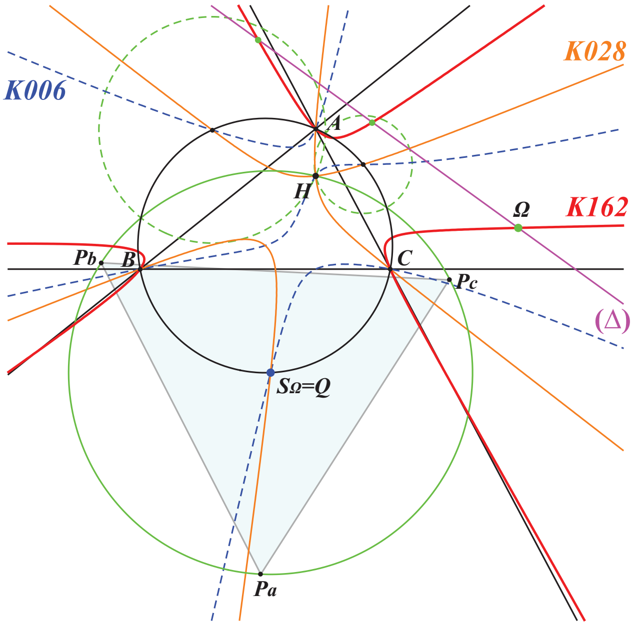

Figure 7: Equilateral triangles $G_aG_bG_c$ and $P_aP_bP_c$ with Ω in $(\Delta) \cap$ K162

Furthermore, for any point Ω on the orthic axis (Δ), the following assertions are equivalent :

1. Ω lies on K162,
2. the circumcircle of $P_aP_bP_c$ passes through H,
3. the center Q of this circle lies on $(\mathcal{O})$ and coincides with S_1, the trilinear pole of the line ΩX_6,
4. the Steiner line of S_1 is perpendicular to the line S_1H.

Note that, in this case, the triangles $G_aG_bG_c$ and $P_aP_bP_c$ are no longer homothetic but remain perspective at S_1. Indeed, the line passing through the isoconjugates of any infinite point under two isoconjugations contains the trilinear pole of the line passing through the two poles of these isoconjugations.

7.2 Ω on (Δ) and on the symmedians

The symmedians of ABC meet (Δ) at K_a, K_b, K_c. The K_a-isoconjugate of the line at infinity is the rectangular hyperbola $(\mathcal{H}_a)$ which is tangent at A to $(\mathcal{O})$ and whose center is the midpoint of AH. The triangle $P_aP_bP_c$ formed with the K_a-isoconjugates of the infinite points of K003 has center A and its circumcircle passes through H. Note that the K_a-isoconjugate of the infinite point I is the intersection of the line AI^* and the perpendicular at H to IH.

This gives a construction of $P_aP_bP_c$ and we find again a classical property of equilateral triangles inscribed in a rectangular hyperbola. Note that $P_aP_bP_c$ and $N_aN_bN_c$ are not homothetic but they are perspective at A. See figure 8.

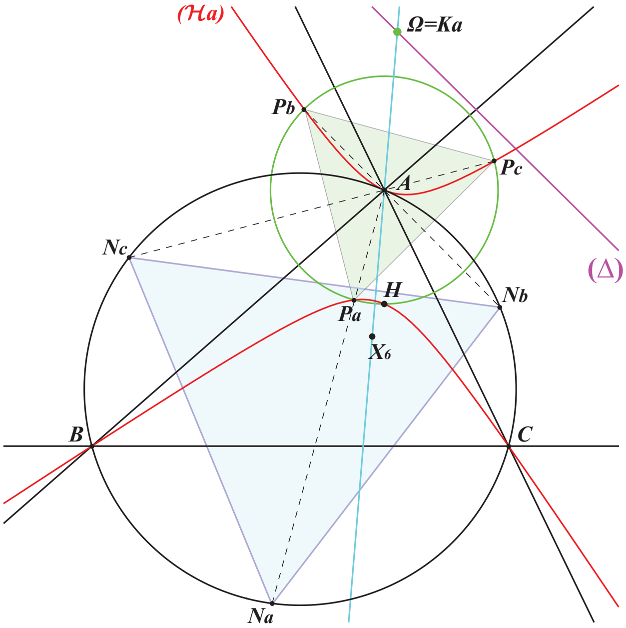

Figure 8: Equilateral triangles $N_aN_bN_c$ and $P_aP_bP_c$ with Ω on (Δ)

8 Summary

Let $(\mathcal{L})$ be the union of three lines making an angle of 60° with one another and let Ω be the pole of an isoconjugation which transforms the infinite points of $(\mathcal{L})$ into the vertices of a triangle $(\mathcal{T})$ obviously inscribed in the circum-conic $(\mathcal{C}_\Omega)$ with perspector Ω.

- if $\Omega = X_6$, $(\mathcal{T})$ is always equilateral with center O and $(\mathcal{C}_\Omega)$ is the circumcircle $(\mathcal{O})$.
- if $\Omega = X_{1989}$, $(\mathcal{T})$ is also always equilateral with center on the perpendicular bisector of OH and sidelines parallel to $(\mathcal{L})$. $(\mathcal{C}_\Omega)$ is the hyperbola $(\mathcal{C}_{1989})$ with eccentricity 2, with focus X_{265} and associated directrix the perpendicular bisector of OH. The parabola with focus X_{265} and directrix the perpendicular bisector of OH is inscribed in $(\mathcal{T})$.
- if Ω lies on the orthic axis (Δ), there are two orientations $(\mathcal{L})$, $(\mathcal{L}')$ giving two equilateral triangles symmetric about the center of the rectangular hyperbola with perspector Ω. One has center H, the other is centered on $(\mathcal{O})$.

- if $\Omega \neq X_6$, $\Omega \neq X_{1989}$ not lying on (Δ), there is one and only one equilateral triangle $(\mathcal{T})$ which is obtained when $(\mathcal{L})$ is the union of three lines parallel to the sidelines of the Morley triangle.

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